14.1/14.2 Partial Derivatives

Goal: To find derivatives of multivariable functions.

Idea: Look at one variable at a time.

Entry Task: Consider

$$f(x,y) = 4xy + y^2 - 3x - 5y$$

1. Plug in y = 1, then find the derivative with respect to x.
2. Do it again with y = 2, …
3. And again with y = 3, …

4. Plug in x = 1, then find the derivative with respect to y.
5. Do it again with x = 2, …
6. And again with x = 3, …

$\boxed{1}$ $f(x,1) = 4x + 1 - 3x - 5 = x - 4$

$\frac{dz}{dx} = 4 + 0 - 3 - 0 = 1$

$\boxed{2}$ $f(x,2) = 4x(2) + (2)^2 - 3x - 5(2)$

$= 8x + 4 - 3x - 10$

$\frac{dz}{dx} = 8 + 0 - 3 + 0 = 5$

$\boxed{3}$ $f(x,3) = 4x(3) + (3)^2 - 3x - 5(3)$

$\frac{dz}{dx} = 12 + 0 - 3 - 0 = 9$

$\boxed{4}$ $f(1,y) = 4(1)y + y^2 - 3(1) - 5y$

$\frac{dz}{dy} = 4 + 2y - 0 - 5$

$\boxed{5}$ $f(2,y) = 4(2)y + y^2 - 3(2) - 5y$

$\frac{dz}{dy} = 8 + 2y - 0 - 5$

$\boxed{6}$ $f(3,y) = 4(3)y + y^2 - 3(3) - 5y$

$\frac{dz}{dy} = 12 + 2y - 0 - 5$

Recall: Definition of derivative

1. Given a function $y = f(x)$.
2. Simplify the general formula for the slope of the secant from x to $x + h$

$$\frac{f(x+h) - f(x)}{h}$$

3. Let $h \to 0$, to get

$$\frac{dy}{dx} = f'(x) = \text{slope of tangent}$$

Partial Derivatives

For multivariable functions:

1. Given $z = f(x, y)$

2a. Simplify (y fixed, x variable)

$$\frac{f(x+h, y) - f(x, y)}{h}$$

3a. Let $h \to 0$, to get

$$\frac{\partial z}{\partial x} = f_x(x, y) \qquad \text{(with respect to x)}$$

Example:

$$f(x, y) = 4xy + y^2 - 3x - 5y$$

$$\frac{f(x+h, y) - f(x, y)}{h} =$$

$$\frac{[4(x+h)y + y^2 - 3(x+h) - 5y] - [4xy + y^2 - 3x - 5y]}{h}$$

$$\frac{\cancel{4xy} + 4hy + \cancel{y^2} \cancel{- 3x} - 3h \cancel{- 5y} \cancel{- 4xy} \cancel{- y^2} \cancel{+ 3x} \cancel{+ 5y}}{h}$$

$$= \frac{4hy - 3h}{h} = 4y - 3$$

$$\frac{\partial z}{\partial x} = f_x(x, y) = 4y - 3$$

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x is variable!

$$4xy + y^2 - 3x - 5y$$

$$4(1)y + 0 - 3 - 0$$

$$4y - 3$$

2b. Simplify (x fixed, y variable)

$$\frac{f(x, y+h) - f(x,y)}{h}$$

3b. Let $h \to 0$, to get

$$\frac{\partial z}{\partial y} = f_y(x,y) \qquad \text{(with respect to x)}$$

y is variable!

$$\boxed{\frac{\partial z}{\partial y} = 4x + 2y - 5}$$

$$4xy + y^2 - 3x - 5y$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$4x(1) + 2y - 0 - 5$$

Example:

$$f(x,y) = 4xy + y^2 - 3x - 5y$$

$$\frac{f(x, y+h) - f(x,y)}{h} =$$

$$\frac{[4x(y+h) + (y+h)^2 - 3x - 5(y+h)] - [4xy + y^2 - 3x - 5y]}{h}$$

$$\frac{\cancel{4xy} + 4xh + \cancel{y^2} + 2yh + h^2 \cancel{-3x} \cancel{-5y} - 5h \cancel{-4xy} \cancel{-y^2} \cancel{+3x} \cancel{+5y}}{h}$$

$$\frac{4xh + 2yh + h^2 - 5h}{h}$$

$$= 4x + 2y + h - 5 \qquad h \to 0$$

How to do partial derivatives:

Step 0: Rewrite powers and simplify like we always do.

Step 1: Identify the desired variable! (Underline it if it helps) Treat all other variable like numbers!

Step 2: Identify the constants terms and the coefficients. "Bring down coefficients"

Step 3: Use the regular one-variable derivative rules.

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

1. $z = 10x^4 + 7xy^3 + 8x^2y^{10}$

$\frac{\partial}{\partial x}$: $10(4x^3) + 7(1)y^3 + 8(2x)y^{10}$

$$\boxed{\frac{\partial z}{\partial x} = 40x^3 + 7y^3 + 16xy^{10}}$$

$\frac{\partial}{\partial y}$: $0 + 7x(3y^2) + 8x^2(10y^9)$

$$\boxed{\frac{\partial z}{\partial y} = 21xy^2 + 80x^2y^9}$$

More examples: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

1. $z = x^3 - y^2 + 3xy^4$

$\frac{\partial}{\partial x}$: $(3x^2) - 0 + 3(1)y^4$

$\boxed{\frac{\partial z}{\partial x} = 3x^2 + 3y^4}$

$\frac{\partial}{\partial y}$: $0 - (2y) + 3x(4y^3)$

$\boxed{\frac{\partial z}{\partial y} = -2y + 12xy^3}$

2. $z = e^{x^2} - \ln(y) + 7$

$\frac{\partial}{\partial x}$: $(e^{x^2} \cdot 2x) - 0 + 0$

$\boxed{\frac{\partial z}{\partial x} = e^{x^2} \cdot (2x) = 2xe^{x^2}}$

$\frac{\partial}{\partial y}$: $0 - \left(\frac{1}{y}\right) + 0$

$\boxed{\frac{\partial z}{\partial y} = -\frac{1}{y}}$

3. $z = (x^2 + 3y)^{10}$

$\frac{\partial}{\partial x}$: $10(x^2+3y)^9 \cdot (2x + 0)$

$$\boxed{\frac{\partial z}{\partial x} = 20x(x^2+3y)^9}$$

$\frac{\partial}{\partial y}$: $10(x^2+3y)^9 \cdot (0 + 3)$

$$\boxed{\frac{\partial z}{\partial y} = 30(x^2+3y)^9}$$

4. $z = \underbrace{x}_{F}\underbrace{y^2 e^x}_{S}$

$F' = 1$

$F = x$

$\frac{\partial}{\partial x}$: product rule! $S = y^2 e^x$

$S' = y^2(e^x)$

$$\underbrace{x}_{F}\, \underbrace{y^2 e^x}_{S'} + \underbrace{(1)}_{F'}\, \underbrace{y^2 e^x}_{S}$$

$$\boxed{\frac{\partial z}{\partial x} = xy^2e^x + y^2e^x}$$

$\frac{\partial}{\partial y}$: $x(2y)e^x$

$$\boxed{\frac{\partial z}{\partial y} = 2xye^x}$$

Interpreting as a rate

Your company produces and sells **two** products (hats and sunglasses)

$x =$ number of hats

$y =$ number of glasses

You find that profit is given by

$$P(x, y) = -3x^2 + 30x - 5y^2 + 130y + 2xy - 100$$

1. Find the partial derivatives.
2. Find and interpret $P_x(5,8)$ and $P_y(5,8)$.

$P_x = -6x + 30 + 2y$

$P_y = -10y + 130 + 2x$

$P_x(5,8) = -6(5) + 30 + 2(8) = 16 = \frac{\partial P}{\partial x}$ ← dollar in profit per ← hats

" The sale of the next hat after (5, 8) will increase profit by about \$16 "

$P_y(5,8) = -10(8) + 130 + 2(5)$
$= -80 + 130 + 10 = 60 = \frac{\partial P}{\partial y}$ ← dollars in profit per ← sunglasses

" The sale of the next sunglasses will increase profit by about \$60. "

3. Estimate the values of

$$\frac{P(5.001,8) - P(5,8)}{0.001} \approx P_x(5,8) = 16$$

x changing

$$\frac{P(5, 8.01) - P(5,8)}{0.01} \approx P_y(5,8) = 60$$

y-changing

Graphical Interpretation

Pretend you are skiing on the surface

$z = f(x, y) = 15 - x^2 - y^2$

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

2. Find and interpret
   $f_x(7,4)$ and $f_y(7,4)$

$f_x = -2x$

$f_y = -2y$

"rate"

$f_x(7,4) = -14$ increase x slightly ⇒ output changes with slope = −14

$f_y(7,4) = -8$ increase y slightly ⇒ output changes with rate = −8.